OSCILLATION CRITERIA FOR NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. The main goal of this article is to study the oscillation criteria of the second-order neutral differential equations on time scales. We give several theorems and related examples to illustrate the applicability of these theorems. Our results extend some recent work in the literature.

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1. Introduction

When delays appear in additional terms involving the highest order derivative of the unknown function in a differential equation, we are dealing with a neutral type differential equation. Neutral functional differential equations have numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see [1]. Recently, many results on oscillation of nonneutral differential equations and neutral functional differential equations have been established. We refer the reader to [2, 3, 4] and the references cited therein.

The theory of time scales is initiated by Hilger [5, 6] in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of the theory of dynamic equations on time scales; see the survey paper by Agarwal et al. [7] and the references cited therein. The books on the subject of time scales, by Bohner and Peterson [8], summarize and organize much of time scale calculus. There are applications of dynamic equations on time scales to quantum...
mechanics, electrical engineering, neural networks, heat transfer, combinatorics, etc.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of ordinary dynamic equations on time scales, we refer the reader to the papers [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Recently Agarwal et al. [24] have established some new oscillation criteria for second-order delay dynamic equations on time scales.

Very recently, some authors studied on existence and behavior of solutions for some integral equations, second order multi objective symmetric programming problem and duality relations and fixed point theorems for nonlinear contractions in [25, 26, 27, 28, 29].

In this paper, we consider second-order nonlinear neutral dynamic equations of the following form:

$$[r(t)((m(t)y(t) + p(t)y(\tau(t))))^\Delta]^\Delta + f_1(t, y(\delta_1(t))) + f_2(t, y(\delta_2(t))) = 0 \quad (1)$$

On a time scale $\mathbb{T}$ satisfying $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$. We recall that a solution of equation (1) is said to be oscillatory on $[t_0, \infty)_\mathbb{T}$ in case it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (1) is said to be oscillatory in case all of its solutions are oscillatory.

Throughout this paper, we assume the followings:

(H1) $\tau(t), \delta_i(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\tau(t) \leq t$ and $\lim_{t \to \infty} \tau(t) = \lim_{t \to \infty} \delta_i(t) = \infty$, $i = 1, 2$,

(H2) $r(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that $\int_{t_0}^{\infty} \frac{1}{r^\gamma(t)} \Delta t = \infty$ and $m(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$,

(H3) $p(t) \in C_{rd}(\mathbb{T}, [0, 1])$ such that $m(\tau(t)) > p(t)$ where $\mathbb{R}^+ = [0, \infty)$,

(f1) $f_i(t, u) : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions such that $uf_i(t, u) > 0$ for all $u \neq 0$ and there exist $q_i(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ $(i = 1, 2)$, $\gamma$ is quotients of odd positive integers $\alpha$ and $\beta$ with $0 < \alpha \leq \beta$ such that $|uf_1(t, u)| \geq q_1(t)|u|^\alpha$, $|uf_2(t, u)| \geq q_2(t)|u|^\beta$.

This paper is organized as follows. After this introduction, we introduce some basic lemmas in Section 2. In Section 3, we present the main results and give an example to illustrate the main results.

2. Some preliminaries

Before stating our main results, we’ll give some lemmas which play an important role in the proof of the main results. Set

$$x(t) := m(t)y(t) + p(t)y(\tau(t)), \quad (2)$$

then the equation (1) becomes

$$(r(t)x^\Delta(t))^\Delta + f_1(t, y(\delta_1(t))) + f_2(t, y(\delta_2(t))) = 0. \quad (3)$$

For $t, T \in \mathbb{T}$ with $t > T$, we define

$$R(t, T) = \int_T^t \frac{1}{(r(s))^\gamma} \Delta s,$$
\[
\beta(t, T) = \begin{cases} \frac{R(\delta(t), T)}{R(\delta(T), T)}, & \delta_1(t) < t; \\ 1, & \delta_1(t) \geq t, \end{cases}
\]

\[
\eta^\alpha(t) = \begin{cases} 1, & \alpha = \gamma; \\ \frac{c}{c^2} \left( \int_0^t \frac{1}{r^\gamma(s)} \Delta s \right)^{\alpha - \gamma}, & \alpha < \gamma; \\ c_1, & \alpha > \gamma, \end{cases}
\]

and

\[Q_1(t) = Q(t) \left( \frac{r^{\frac{1}{\gamma}}(t) R(t, T)}{r^{\frac{1}{\gamma}}(t) R(t, T) + \mu(t)} \right)^{\alpha} \eta^\gamma(t),\]

where \(Q(t)\) will be defined as Lemma 2.2. For \(D = \{(t, s) \in \mathbb{T}^2 : t \geq s \geq 0\}\), we define

\[
\mathcal{H} = \{H(t, s) \in C_{rd}^1(D, [0, \infty)) : H(t, t) = 0, H(t, s) > 0 \text{ and } H(t, s) \geq 0 \text{ for } t > s \geq 0\}
\]

and

\[C(t, s) = H(t, t) z(s) + H(t, s) z(t), \quad H(t, s) \in \mathcal{H}, \]

where \(z \in C_{rd}^1(D, [0, \infty))\) is to be given Theorem 3.2 and Theorem 3.3, and 

\[z(t) = \max_{\mathcal{H}} \{z(t)\}, \]

**Lemma 2.1.** Assume that \((H_1) - (H_3)\) and \((1)\) are satisfied. If equation \((1)\) has a nonoscillatory solutions \(y\) on \([t_0, \infty)_\mathbb{T}\), and \(x\) is defined as in \((2)\), then there exists a \(T \in \mathbb{T}\) sufficiently large such that \(x(t) > 0, x^\Delta(t) > 0, (r(t) (x^\Delta(t))^\gamma)^\Delta < 0, x(t) \geq r^{\frac{1}{\gamma}}(t) x^\Delta(t) R(t, T), x(\delta(t)) \geq \beta(t, T) x(t) \) for \(t \in [T, \infty)_\mathbb{T}\).

**Proof.** If \(y(t)\) is an eventually positive solution of \((1)\), then there exist a \(T \in [t_0, \infty)_\mathbb{T}\) such that

\[y(t) > 0, \quad y(\tau(t)) > 0, \quad y(\delta_i(t)) > 0, \quad \text{for } t \geq T, \quad i = 1, 2. \tag{4}\]

From \((2), (4)\) and \((H_2)\), \(x(t) > 0\). Also by \((3)\) and \((H_3)\), we have

\[(r(t)(x^\Delta(t))^\gamma)^\Delta = -f_1(t, y(\delta_1(t))) - f_2(t, y(\delta_2(t))) \leq -q_1(t) y^{\alpha}(\delta_1(t)) - q_2(t) y^{\beta}(\delta_2(t)) < 0, \quad \text{for } t \geq T,\]

which implies that \((r(t)(x^\Delta(t))^\gamma)^\gamma\) is decreasing on \([T, \infty)_\mathbb{T}\).

We claim that \((r(t)(x^\Delta(t))^\gamma)^\gamma > 0 \) on \([T, \infty)_\mathbb{T}\). Assume not, there is a \(t_1 \in [T, \infty)_\mathbb{T}\) such that \((r(t_1)(x^\Delta(t_1))^\gamma)^\gamma = c < 0\). Since \((r(t)(x^\Delta(t))^\gamma)^\gamma\) is decreasing on \([T, \infty)_\mathbb{T}\),

\[r(t)(x^\Delta(t))^\gamma \leq r(t_1)(x^\Delta(t_1))^\gamma = c \quad \text{for } t \geq t_1. \]

So, we have

\[x^\Delta(t) \leq \frac{c^\gamma}{r^\gamma(t)}.\]

Integrating the above inequality from \(t_1\) to \(t\), by \((H_2)\), we get

\[x(t) \leq x(t_1) + c \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s \to -\infty \quad (t \to \infty)\]
and this contradicts the fact that \( x(t) > 0 \) for all \( t \geq T \). Thus, we have \( r(t)(x^\Delta(t))^\gamma > 0 \) on \([T, \infty)_T\) and so \( x^\Delta(t) > 0 \) on \([T, \infty)_T\). From \( r(t)(x^\Delta(t))^\gamma \) is decreasing on \([T, \infty)_T\), we have

\[
 x(t) = x(T) + \int_T^t x^\Delta(s) \Delta s \\
 = x(T) + \int_T^t \left( \frac{r(s)(x^\Delta(s))^\gamma}{r^\Delta(s)} \right) \Delta s \\
 > \int_T^t \left( \frac{r(s)(x^\Delta(s))^\gamma}{r^\Delta(s)} \right) \Delta s \\
 > \left( r(t)(x^\Delta(t))^\gamma \right)^{\frac{1}{\gamma}} \int_T^t \frac{1}{r^\Delta(s)} \Delta s \\
 = r^{\frac{1}{\gamma}}(t)x^\Delta(t)R(t, T).
\]

Now, we will show that

\[ x(\delta_i(t)) \geq \beta(t, T)x(t). \]

We consider two cases which \( \delta_i(t) < t \) and \( \delta_i(t) \geq t \), respectively.

**Case 1.** \( \delta_i(t) < t \). Since \( r(t)(x^\Delta(t))^\gamma \) is decreasing on \([T, \infty)_T\), we have

\[
 x(t) - x(\delta_i(t)) = \int_{\delta_i(t)}^t \left( \frac{r(s)(x^\Delta(s))^\gamma}{r^\Delta(s)} \right) \Delta s \\
 \leq \left( r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma \right)^{\frac{1}{\gamma}} \int_{\delta_i(t)}^t \frac{1}{r^\Delta(s)} \Delta s.
\]

By dividing \( x(\delta_i(t)) \), it follows that

\[
 \frac{x(t)}{x(\delta_i(t))} \leq 1 + \frac{\left( r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma \right)^{\frac{1}{\gamma}} \int_{\delta_i(t)}^t \frac{1}{r^\Delta(s)} \Delta s}{x(\delta_i(t))}.
\]

Since \( \delta_i(t) \geq T \) for \( t \in [T, \infty)_T \), we have

\[
 x(\delta_i(t)) \geq \int_T^{\delta_i(t)} \frac{r(s)(x^\Delta(s))^\gamma}{r^\Delta(s)} \Delta s \geq \left( r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma \right)^{\frac{1}{\gamma}} \int_T^{\delta_i(t)} \frac{1}{r^\Delta(s)} \Delta s,
\]

which implies that

\[
 \frac{(r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{\frac{1}{\gamma}}}{x(\delta_i(t))} \leq \frac{1}{\int_T^{\delta_i(t)} \frac{1}{r^\Delta(s)} \Delta s}.
\]

Thus, we get

\[
 \frac{x(t)}{x(\delta_i(t))} \leq 1 + \frac{\int_{\delta_i(t)}^t \frac{1}{r^\Delta(s)} \Delta s}{\int_T^{\delta_i(t)} \frac{1}{r^\Delta(s)} \Delta s} = \frac{\int_{\delta_i(t)}^t \frac{1}{r^\Delta(s)} \Delta s}{\int_T^{\delta_i(t)} \frac{1}{r^\Delta(s)} \Delta s} = \frac{R(t, T)}{R(\delta_i(t), T)}.
\]
Assume that conditions (6) hold. If (5) is satisfied, then there exists a $t \geq 0$ such that

\[ x(t) \geq y(t) \geq 0 \quad \text{for} \quad t \geq 0, \]

and (1) is satisfied. If $x(t) \geq 0$, then there exists a $T > t$ such that

\[ x(T) > 0 \quad \text{for} \quad t \geq T, \]

and (2) then there exists a $t \geq 0$ such that

\[ x(t) \geq y(t) \geq 0 \quad \text{for} \quad t \geq 0. \]

Case 2. Let $y(t)$ be a nonoscillatory solution of (1). From Lemma 2.1 and (6), we have

\[ x(\delta_1(t)) \geq \beta(t,T)x(t). \]

Proof. From Lemma 2.1 and (6), we have

\[ (r(t)x(\Delta(t)))^\Delta = (r(t)x(\Delta(t)))^\alpha + Q(t)x^\beta(t) \leq 0 \quad \text{for} \quad t \geq T, \]

where

\[ Q(t) = q_1(t)\left(\frac{1}{m(\delta_1(t))}\left[1 - \frac{p(\delta_1(t))}{m(\tau(\delta_1(t)))}\right]\right)^\alpha \beta(t,T) + q_2(t)\left(\frac{1}{m(\delta_2(t))}\left[1 - \frac{p(\delta_2(t))}{m(\tau(\delta_2(t)))}\right]\right)^\beta \beta(t,T) > 0. \]

Proof. If $y(t)$ is an eventually positive solution of (1) then there exists a $T \in [t_0, \infty)$ such that

\[ y(t) > 0, \quad y(\tau(t)) > 0, \quad y(\delta_1(t)) > 0, \quad y(\tau(\delta_1(t))) > 0 \quad \text{for} \quad t \geq T, \quad i = 1, 2. \]

From Lemma 2.1 and (6), we have

\[ (r(t)x(\Delta(t)))^\Delta \leq -q_1(t)y_\alpha(\delta_1(t)) - q_2(t)y_\beta(\delta_2(t)) \]

\[ \leq -q_1(t)\left(\frac{1}{m(\delta_1(t))}\left[1 - \frac{p(\delta_1(t))}{m(\tau(\delta_1(t)))}\right]\right)^\alpha x_\alpha(\delta_1(t)) + q_2(t)\left(\frac{1}{m(\delta_2(t))}\left[1 - \frac{p(\delta_2(t))}{m(\tau(\delta_2(t)))}\right]\right)^\beta x_\beta(\delta_2(t)). \]
Thus,
\[
(r(t)(x^\Delta(t))^\Delta + Q(t)x^\alpha(t)) \leq 0. 
\] (7)

**Lemma 2.3.** [9] Let \( g(u) = Bu - Au^\frac{\gamma+1}{\gamma} \), where \( A > 0 \) and \( B \) are constants, \( \gamma \) is a quotient of odd positive integers. Then \( g \) attains its maximum value on \( \mathbb{R} \) at \( u^* = \left( \frac{B^\gamma}{A^{\gamma+1}} \right)^\frac{1}{\gamma} \) and
\[
\max_{u \in \mathbb{R}} g(u^*) = \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}. 
\] (8)

### 3. Main Results

In this section, we state and prove the main oscillation results for the equations (1).

**Theorem 3.1.** Assume that \((H_1) - (H_3)\) and (1) are satisfied. If
\[
\int_{t_0}^{\infty} Q_1(s)\Delta s = \infty, 
\] (9)
then every solution of (1) oscillates.

**Proof.** Assume the contrary and let \( y \) be a nonoscillatory solution of (1). Without loss of generality, we may assume that
\[
y(t) > 0, \; y(\tau(t)) > 0, \; y(\tau(\tau(t))) > 0, \; y(\delta_i(t)) > 0, \; i = 1, 2 \; \text{for} \; t \geq T. \tag{10}
\]
We define
\[
w(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)} \; \text{for} \; t \geq T, \tag{11}
\]
where
\[
x^{[1]}(t) := (r(t)(x^\Delta(t))^\gamma)(t) \; \text{and} \; x^{[2]}(t) := (x^{[1]}(t))^\Delta.
\]
Then, \( w(t) > 0 \; \text{for} \; t \geq T. \) Since Lemma 2.1 and (2), there exists a \( T \geq t_0 \) such that
\[
x(t) > 0, \; x^{[1]}(t) > 0 \; \text{and} \; x^{[2]}(t) < 0 \; \text{for} \; t \geq T. \tag{12}
\]
From Lemma 2.2, we get

\[ w^\Delta(t) = \frac{x^{[2]}(t)x^\gamma(t) - (x^\gamma(t))^\Delta x^{[1]}(t)}{x^\gamma(t)(x^\sigma(t))^\gamma} \leq -Q(t) \frac{x^\alpha(t)}{(x^\sigma(t))^\gamma} - \frac{(x^\gamma(t))^\Delta x^{[1]}(t)}{x^\gamma(t)(x^\sigma(t))^\gamma}. \]  

(13)

By the Ptzsche chain rule, if \( x^\Delta(t) > 0 \) and \( \gamma > 1 \), then we get

\[ (x^\gamma(t))^\Delta = \gamma \int_0^1 [x(t) + \mu(t)hx^\Delta(t)]^{\gamma-1}x^\Delta(t)dh \]
\[ = \gamma \int_0^1 [(1-h)x(t) + hx^\sigma(t)]^{\gamma-1}x^\Delta(t)dh \]
\[ \geq \gamma \int_0^1 (x(t))^{\gamma-1}x^\Delta(t)dh \]
\[ = \gamma(x(t))^{\gamma-1}x^\Delta(t). \]

Again by the Ptzsche chain rule, if \( x^\Delta(t) > 0 \) and \( 0 < \gamma \leq 1 \), then we have

\[ (x^\gamma(t))^\Delta = \gamma \int_0^1 [x(t) + \mu(t)hx^\Delta(t)]^{\gamma-1}x^\Delta(t)dh \]
\[ = \gamma \int_0^1 [(1-h)x(t) + hx^\sigma(t)]^{\gamma-1}x^\Delta(t)dh \]
\[ \geq \gamma \int_0^1 (x^\sigma(t))^{\gamma-1}x^\Delta(t)dh \]
\[ = \gamma(x^\sigma(t))^{\gamma-1}x^\Delta(t). \]

Since \( x(t) \) is increasing and \( x^{[1]}(t) \) is decreasing, for \( \gamma > 1 \), we get

\[ \frac{((x(t))^\gamma)^\Delta x^{[1]}(t)}{x^\gamma(t)(x^\sigma(t))^\gamma} \geq \gamma(x(t))^{\gamma-1}x^\Delta(t)(x^{[1]}(t))^\sigma \]
\[ \geq \gamma((x^{[1]}(t))^\sigma)^{\sigma} \]
\[ \geq \gamma((x^{[1]}(t))^\sigma)^{\frac{\sigma}{2}}(x^{[1]}(t))^\sigma \]
\[ \geq \gamma \left( \frac{1}{r^2(t)x(\sigma(t))(x(\sigma(t)))^\gamma} \right)^{\frac{\sigma}{2}}(x^{[1]}(t))^\sigma \]
\[ = \gamma \frac{1}{r^{\frac{\sigma}{2}}(t)}(w^\sigma(t))^{\frac{\sigma}{2}+1}. \]  

(14)

Also for \( 0 < \gamma \leq 1 \), we have

\[ \frac{((x(t))^\gamma)^\Delta x^{[1]}(t)}{x^\gamma(t)(x^\sigma(t))^\gamma} \geq \gamma(x(\sigma(t)))^{\gamma-1}x^\Delta(t)(x^{[1]}(t))^\sigma \]
\[ \geq \gamma \left( \frac{1}{r^2(t)x(\sigma(t))(x(\sigma(t)))^\gamma} \right)^{\frac{\sigma}{2}}(x^{[1]}(t))^\sigma \]
\[ = \gamma \frac{1}{r^{\frac{\sigma}{2}}(t)}(w^\sigma(t))^{\frac{\sigma}{2}+1}. \]
\[ \frac{\gamma(x^{[1]}(t))^{\frac{1}{\gamma}}(x^{[1]}(t))^\sigma}{r^\tau(t)x(x(t))^{\gamma}} \geq \frac{\gamma((x^{[1]}(t))^{\sigma})^{\frac{1}{\gamma}}(x^{[1]}(t))^{\sigma}}{r^\tau(t)x(x(t))^{\gamma}} \]
\[ = \frac{1}{r^\tau(t)}(w^\sigma(t))^{\frac{1}{\gamma}+1}. \]  

(15)

Together (14) and (15), we obtain
\[ \frac{((x(t))^{\gamma})\Delta x^{[1]}(t)}{x^\gamma(t)x^\sigma(t)^{\gamma}} \geq \gamma \frac{1}{r^\tau(t)}(w^\sigma(t))^{\frac{1}{\gamma}+1} \text{ for } \gamma > 0. \]

(16)

Substituting (16) in (13), we get
\[ w^\Delta(t) \leq -Q(t)\frac{x^\sigma(t)}{(x(t))^{\gamma}} - \gamma \frac{1}{r^\tau(t)}(w^\sigma(t))^{\frac{1}{\gamma}+1}. \]

(17)

Since \( x^\sigma(t) = x(t) + \mu(t)x^\Delta(t) \), we have
\[ \frac{x^\sigma(t)}{x(t)} = 1 + \mu(t)\frac{x^\Delta(t)}{x(t)} = 1 + \frac{\mu(t)}{r^\tau(t)} \frac{x^{[1]}(t)}{x(t)}. \]

(18)

Since \( x^{[1]}(t) \) is decreasing, we get
\[ x(t) = x(T) + \int^t_T (x^{[1]}(s))^{\frac{1}{\gamma}} \left( \frac{1}{r(s)} \right)^{\frac{1}{\gamma}} \Delta s \]
\[ > (x^{[1]}(t))^{\frac{1}{\gamma}} \int^t_T \left( \frac{1}{r(s)} \right)^{\frac{1}{\gamma}} \Delta s. \]

Therefore,
\[ \frac{x(t)}{(x^{[1]}(t))^{\frac{1}{\gamma}}} \geq \int^t_T \left( \frac{1}{r(s)} \right)^{\frac{1}{\gamma}} \Delta s = R(t, T), \text{ for } t \geq T. \]

(19)

Thus we have
\[ \frac{x^\sigma(t)}{x(t)} = 1 + \frac{\mu(t)}{r^\tau(t)} \frac{x^{[1]}(t)}{x(t)} \leq \frac{r^\tau(t)R(t, T) + \mu(t)}{r^\tau(t)R(t, T)} \text{ for } t \geq T \]
and so
\[ \frac{x(t)}{x^\sigma(t)} \geq \frac{r^\tau(t)R(t, T)}{r^\tau(t)R(t, T) + \sigma(t) - t} \text{ for } t \geq T. \]

Thus, for \( t \geq T \), we get
\[ \frac{x^\alpha(t)}{(x(t))^\gamma} = \left( \frac{x(t)}{x(x(t))^{\alpha}} \right)^{\gamma} \frac{1}{(x(x(t)))^{\gamma-\alpha}} \]
\[ \left[ \frac{r^\gamma(t)R(t, T)}{r^\gamma(t)R(t, T) + \mu(t)} \right] (x(\sigma(t)))^{\alpha - \gamma}. \] (20)

Consider the following cases.

**Case 1.** Let \( \alpha < \gamma \). Since \( x^{[1]}(t) \) is positive and decreasing, it follows from Lemma 2.1 that \( x^{[1]}(t) \leq x^{[1]}(T) = c \) for \( t \geq T \). So,

\[ x^{\Delta}(t) \leq \frac{c^\gamma}{r^\gamma(t)}. \] (21)

This implies that,

\[ x(\sigma(t)) = x(t_2) + \int_{T}^{\sigma(t)} x^{\Delta}(s) \Delta s \leq x(t_2) + \frac{c^\gamma}{r^\gamma(s)} \Delta s. \] (22)

Since \( \alpha < \gamma \), we get

\[ x^{\alpha-\gamma}(t) > c_2^{\alpha-\gamma} \left( \int_{T}^{\sigma(t)} \frac{1}{r^\gamma(s)} \Delta s \right)^{\alpha-\gamma}, \] (23)

where

\[ c_2 = \left( \frac{1}{c} \right)^\gamma. \] (24)

**Case 2.** Let \( \alpha = \gamma \). So, \( (x^\sigma(t))^{\alpha-\gamma} = 1 \).

**Case 3.** Let \( \alpha > \gamma \). In this case, since \( x^{\Delta}(t) > 0 \), there exist \( t_2 \geq t_1 \) such that \( x^\sigma(t) > x(t) > c > 0 \). This implies that \( (x^\sigma(t))^{\alpha-\gamma} > c_1 \), where \( c_1 = c^{\alpha-\gamma} \).

Combining these three cases and using the definition of \( \eta^\sigma \), we conclude that

\[ (x^\sigma(t))^{\alpha-\gamma} \geq \eta^\sigma(t). \]

Hence,

\[ \frac{x^\alpha(t)}{(x(\sigma(t)))^\gamma} \geq \left( \frac{r^\gamma(t)R(t, T)}{r^\gamma(t)R(t, T) + \mu(t)} \right)^\alpha \eta^\sigma(t). \] (25)

Substituting (25) to (17), we obtain,

\[ w^{\Delta}(t) \leq -Q(t) \left( \frac{r^\gamma(t)R(t, T)}{r^\gamma(t)R(t, T) + \mu(t)} \right)^\alpha \eta^\sigma(t) - \gamma \frac{1}{r^\gamma(t)} (w^\sigma(t))^\frac{1}{\gamma+1}, \]

\[ w^{\Delta}(t) + Q_1(t) + \gamma \frac{1}{r^\gamma(t)} (w^\sigma(t))^\frac{1}{\gamma+1} \leq 0 \] (26)

and so

\[ -w^{\Delta}(t) \geq Q_1(t) + \gamma \frac{1}{r^\gamma(t)} (w^\sigma(t))^\frac{1}{\gamma+1} > Q_1(t), \text{ for } t \geq T. \] (27)
It follows from the definition of $x^{[1]}(t)$ that

$$x^{\Delta}(t) = \left(\frac{x^{[1]}(t)}{r(t)}\right)^{1/\gamma}. \quad (28)$$

Integrating (28) from $T$ to $t$, we obtain

$$x(t) = x(T) + \int_T^t \left(\frac{1}{r(s)}x^{[1]}(s)\right)^{1/\gamma} \Delta s, \quad \text{for } t \geq T. \quad (29)$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get

$$x(t) \geq x(T) + \left(\frac{x^{[1]}(t)}{r(T)}\right)^{1/\gamma} \int_T^t \frac{1}{r(s)} \Delta s \quad \text{for } t \geq T.$$

Thus, we have

$$\frac{(x^{[1]}(t))^{1/\gamma}}{x(t)} \leq \left(\int_T^t \frac{1}{r(s)} \Delta s\right)^{-1}$$

and

$$w(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq \left(\int_T^t \frac{1}{r(s)} \Delta s\right)^{-\gamma} \quad \text{for } t \in [T, \infty)_T,$$

which implies, in view of $(H_2)$, that

$$\lim_{t \to \infty} w(t) = 0.$$

Integrating (27) from $T$ to $\infty$ and using the fact that $\lim_{t \to \infty} w(t) = 0$ we obtain

$$w(T) \geq \int_T^\infty Q_1(s) \Delta s,$$

which contradicts (9). The proof is completed. \(\square\)

**Theorem 3.2.** Assume that $(H_1) - (H_3)$ and (1) are satisfied. If there exists a positive rd-continuous $\Delta$-differentiable function $z(t)$ such that

$$\lim_{t \to \infty} \sup_{t \geq T} \int_T^t \left[ z(s)Q_1(s) - \frac{r(s)(z^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)} \right] = \infty, \quad (30)$$

then every solution of (1) oscillates.

**Proof.** Assume the contrary and let $y$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that

$$y(t) > 0, \quad y(\tau(t)) > 0, \quad y(\tau(\tau(t))) > 0, \quad y(\delta_i(t)) > 0, \quad i = 1, 2 \quad \text{for } t \geq T. \quad (31)$$

Let $w$ be defined as in (11), then $w(t) > 0$ for $t \geq T$. Using (26), the following inequality is true:

$$w^{\Delta}(t) \leq -Q_1(t) - \frac{1}{r^{\gamma}(t)}(w^{\sigma}(t))^{1/\gamma+1} \quad \text{for } t \geq T. \quad (32)$$
Multiplying (32) by $z(t)$ and integrating from $T$ to $t$, we obtain
\[
\int_T^t z(s)Q_1(s)\Delta s \leq - \int_T^t z(s)w^\Delta(s)\Delta s - \int_T^t \frac{\gamma z(s)}{r^\gamma(s)} (w^\sigma(s))^{\frac{1}{\gamma}+1} \Delta s. \tag{33}
\]
Integration by parts, we get
\[
- \int_T^t z(s)w^\Delta(s)\Delta s = -z(t)w(t) + z(T)w(T) + \int_T^t z^\Delta(s)w^\sigma(s)\Delta s \\
\quad \leq z(T)w(T) + \int_T^t z^\Delta(s)w^\sigma(s)\Delta s.
\]
It follows that
\[
\int_T^t z(s)Q_1(s) \leq z(T)w(T) + \int_T^t z^\Delta(s)w^\sigma(s)\Delta s - \int_T^t \frac{\gamma z(s)}{r^\gamma(s)} (w^\sigma(s))^{\frac{1}{\gamma}+1} \Delta s.
\]
Setting $B = z^\Delta(s)$, $A = \frac{\gamma z(s)}{r^\gamma(s)}$, $u = w^\sigma(s)$ and using Lemma 2.3, we get
\[
\int_T^t z(s)Q_1(s)\Delta s \leq z(T)w(T) + \int_T^t \frac{\gamma}{(\gamma + 1)^{\gamma+1}} (z^\Delta(s))^{\gamma+1}r(s) \quad \gamma z\gamma(s)
\]
and so
\[
\int_T^t \left[ z(s)Q_1(s) - \frac{r(s)(z^\Delta(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1}z\gamma(s)} \right] \Delta s \leq z(T)w(T), \tag{34}
\]
which contradicts condition (30). Then every solution of (1) oscillates. The proof is completed. \hfill \Box

**Theorem 3.3.** Assume that $(H_1)-(H_3)$ and (1) are satisfied. Suppose that $z(t)$ is defined as in Theorem 3.2, $H \in R$ and for $t > s$,
\[
\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_T^t \left[ H(t, s)z(s)Q_1(s) - \frac{r(s)c^{\gamma+1}(t, s)}{(\gamma + 1)^{\gamma+1}z\gamma(s)}H^\gamma(t, s) \right] \Delta s = \infty, \quad \tag{35}
\]
is satisfied. Then every solution of (1) oscillates.

**Proof.** Assume the contrary and let $y$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that
\[
y(t) > 0, \quad y(r(t)) > 0, \quad y(r(r(t))) > 0, \quad y(\delta_i(t)) > 0, \quad i = 1, 2 \quad \text{for} \quad t \geq T. \tag{36}
\]
Let $w$ be defined as in (11), then $w(t) > 0 \quad \text{for} \quad t \geq T$. Multiplying (32) by $z(t)H(t, u)$ and integrating from $T$ to $t$, we obtain
\[
\int_T^t H(t, s)z(s)Q_1(s)\Delta s \quad \leq \quad - \int_T^t H(t, s)z(s)w^\Delta(s)\Delta s \\
\quad - \int_T^t \frac{\gamma z(s)}{r^\gamma(s)} (w^\sigma(s))^{\frac{1}{\gamma}+1} \Delta s, \quad \tag{37}
\]
for \( u = t \). Integration by parts we get
\[
- \int_T^t H(t, s)z(s)w^\Delta(s)\Delta s = H(t, T)z(T)w(T) + \int_T^t (H(t, s)z(s))w^\Delta(s)\Delta s.
\]
Thus, we have
\[
\int_T^t H(t, s)z(s)Q_1(s)\Delta s
\leq \quad H(t, T)z(T)w(T) + \int_T^t (H^\Delta(t, s)z(s) + H^\eta(t, s)z^\Delta(s))w^\sigma(s)\Delta s
- \int_T^t \frac{\gamma(s)H(t, s)}{r^\Delta(s)}(w^\sigma(s))^{\frac{1}{\gamma}}\Delta s
= \quad H(t, T)z(T)w(T) + \int_T^t C(t, s)w^\sigma(s)\Delta s - \int_T^t \frac{\gamma(s)H(t, s)}{r^\Delta(s)}(w^\sigma(s))^{\frac{\gamma+1}{\gamma}}\Delta s.
\]
Setting \( B = C(t, s), \quad A = \frac{H(t, s)\gamma(s)}{r^\Delta(s)}, \) \( u = w^\sigma(s) \) and using Lemma 2.3, we have
\[
\int_T^t H(t, s)z(s)Q_1(s)\Delta s \leq H(t, T)z(T)w(T) + \int_T^t \frac{\gamma(s)}{(\gamma + 1)^{\gamma+1}} H^\theta(t, s)^{\gamma^2} \Delta s
\]
and
\[
\int_T^t \left[ H(t, s)z(s)Q_1(s) - \frac{r(s)C^{\gamma+1}(t, s)}{(\gamma + 1)^{\gamma+1}z^\gamma(s)H^\gamma(t, s)} \right] \Delta s \leq H(t, T)z(T)w(T), \quad (38)
\]
which contradicts condition (30). Then every solution of (1) oscillates. The proof is completed.

4. Example

**Example 4.1.** Let \( T \) be any time scales and we consider the following second order neutral dynamic equation
\[
\left( \left( \left( t^\gamma(t) + \frac{1}{2} \right) \frac{\Delta}{2} \right) \right) \Delta + \frac{\sigma(t)}{2t^3} y^\gamma(t-\frac{1}{2}) + \frac{\sigma(t)}{4t^5} y^\gamma(t-\frac{1}{4}) = 0, \quad (39)
\]
where \( t \in [2, \infty)_\gamma \).

(H1) \( \tau(t) = \frac{1}{2} \leq \delta_1(t) = t - \frac{1}{2} < t, \quad \delta_2(t) = t - \frac{1}{4} < t, \lim_{t \to \infty} \tau(t) = \lim_{t \to \infty} \delta_1(t) = \infty, \quad i = 1, 2. \)

(H2) \( \tau(t) = \frac{1}{2} \in C_{rd}(T, \mathbb{R}^+), \quad r(t) = 1 \in C_{rd}(T, \mathbb{R}^+), \) \( \int_2^\infty \frac{1}{r^\gamma(t)} \Delta t = \int_2^\infty \Delta t = \lim_{a \to \infty} \int_a^\infty \Delta t = \lim_{a \to \infty} a - 2 = \infty, \) \( m(t) = t \in C_{rd}(T, \mathbb{R}^+), \) \( p(t) = \frac{1}{2} \in C_{rd}(\{0, 1\}) \) that \( m(\tau(t)) = \frac{1}{2} \geq 1 \geq \frac{1}{2} = p(t) \) for \( t \geq 2. \)

(H3) \( q_1(t) = \frac{\sigma(t)}{2(2t-3)^2}, \quad \alpha = \frac{1}{5}, \quad q_2(t) = \frac{\sigma(t)}{2(4t-5)^2}, \quad \beta = \frac{1}{5}, \) that \( q_i(t) \in C_{rd}(T, \mathbb{R}^+). \)
Since
\[ Q(t) = q_1(t) \left( \frac{1}{m(\delta_1(t))} \left[ 1 - \frac{p(\delta_1(t))}{m(\tau(\delta_1(t)))} \right] \right)^\alpha \beta^\alpha(t, T) \\
+ q_2(t) \left( \frac{1}{m(\delta_2(t))} \left[ 1 - \frac{p(\delta_2(t))}{m(\tau(\delta_2(t)))} \right] \right)^\beta \beta^\beta(t, T) \\
= \frac{(\sigma(t))^2}{(2t - 3)^2} \left( \frac{1}{t - \frac{1}{2}} \left[ 1 - \frac{1}{2 \cdot \frac{1}{2}(t - \frac{1}{2})} \right] \right)^\frac{1}{2} \left( \frac{\delta_1(t) - T}{t - T} \right)^\frac{1}{2} \\
+ \frac{(\sigma(t))^2}{(4t - 5)^2} \left( \frac{1}{t - \frac{1}{4}} \left[ 1 - \frac{1}{2 \cdot \frac{1}{4}(t - \frac{1}{4})} \right] \right)^\frac{1}{2} \left( \frac{\delta_2(t) - T}{t - T} \right)^\frac{1}{2} \\
= \frac{(\sigma(t))^2}{(2t - 3)^2} \frac{2^{\frac{1}{2}}}{(2t - 1)^\frac{1}{2}} \left( \frac{\delta_1(t) - T}{t - T} \right)^\frac{1}{2} \\
+ \frac{(\sigma(t))^2}{(4t - 5)^2} \frac{4^{\frac{1}{2}}}{(4t - 1)^\frac{1}{2}} \left( \frac{\delta_2(t) - T}{t - T} \right)^\frac{1}{2},
\]
we get
\[ Q_1(t) = Q(t) \left( \frac{r^\pm(t)R(t, T)}{r^\pm(t)R(t, T) + \mu(t)} \right)^\alpha \eta^\alpha(t), \]
where
\[ \left( \frac{r^\pm(t)R(t, T)}{r^\pm(t)R(t, T) + \mu(t)} \right)^\alpha = \left( \frac{t - T}{t - T + \sigma(t) - t} \right)^\frac{1}{2} = \left( \frac{t - T}{\sigma(t) - T} \right)^\frac{1}{2}. \]
and
\[ \eta^\alpha(t) = c_2^\alpha \gamma \left( \int_T^{\sigma(t)} \frac{1}{r^\gamma(s)} \Delta s \right)^{\alpha - \gamma} = c_2^{-\frac{3}{2}} (\sigma(t) - T)^{-\frac{3}{2}}. \]
Thus we get
\[ Q(t) \left( \frac{t - T}{\sigma(t) - T} \right)^\frac{1}{2} = \frac{(\sigma(t))^2}{(2t - 1)^\frac{1}{2}} \left( \frac{\delta_1(t) - T}{t - T} \right)^\frac{1}{2} \left( \frac{t - T}{\sigma(t) - T} \right)^\frac{1}{2} \\
+ \frac{(\sigma(t))^2}{(4t - 1)^\frac{1}{2}} \left( \frac{\delta_2(t) - T}{t - T} \right)^\frac{1}{2} \left( \frac{t - T}{\sigma(t) - T} \right)^\frac{1}{2} \\
> \frac{(\sigma(t))^2}{(2t - 1)^\frac{1}{2}} \left( \frac{\delta_1(t) - T}{\sigma(t) - T} \right)^\frac{1}{2} + \frac{(\sigma(t))^2}{(4t - 1)^\frac{1}{2}} \left( \frac{\delta_2(t) - T}{\sigma(t) - T} \right)^\frac{1}{2}. \]
Archimedes property says if \( x \) and \( y \) are real numbers with \( x > 0 \), there exists a natural \( n \) such that \( nx > y \). So, for \( t > T \) and \( \delta_1(t) > T \) there exist constants \( k_1, k_2 > 0 \) sufficiently large that
\[ Q_1(t) > \frac{1}{c_2^2 (\sigma(t) - T)^\frac{3}{2}} \left( \frac{(\sigma(t))^2}{2^\frac{1}{2}} \left( \frac{t}{k_1 \sigma(t)} \right)^\frac{1}{2} + \frac{(\sigma(t))^2}{4^\frac{1}{2}} \left( \frac{t}{k_2 \sigma(t)} \right)^\frac{1}{2} \right) \]
\[
> \frac{(\sigma(t))^\frac{3}{2}2^{\frac{1}{2}}}{2^{\frac{1}{2}}t^{\frac{1}{2}}} \left( \frac{t}{k_1\sigma(t)} \right)^{\frac{1}{4}} \frac{1}{c_2^2(\sigma(t))^{\frac{1}{2}}} + \frac{(\sigma(t))^\frac{3}{2}4^{\frac{1}{2}}}{4^{\frac{1}{2}}t^{\frac{1}{2}}} \left( \frac{t}{k_2\sigma(t)} \right)^{\frac{1}{4}} \frac{1}{c_2^2(\sigma(t))^{\frac{1}{2}}}
\]
\[
= \frac{1}{2^{\frac{1}{2}}k_1^\frac{1}{4}c_2^2t^{\frac{1}{2}}} + \frac{1}{4^{\frac{1}{2}}k_2^\frac{1}{4}c_2^2t^{\frac{1}{2}}}
\]

Let \( z(t) = 1 \), thus we obtain,
\[
\lim_{t \to \infty} \sup_{T} \int_{t}^{T} \left[ z(s)Q_1(s) - \frac{r(s)(z(\Delta)(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1}z(\gamma)(s)} \right]
\geq \lim_{t \to \infty} \sup_{T} \int_{t}^{T} \left[ \frac{1}{2^{\frac{1}{2}}k_1^\frac{1}{4}c_2^2s^{\frac{1}{2}}} + \frac{1}{4^{\frac{1}{2}}k_2^\frac{1}{4}c_2^2s^{\frac{1}{2}}} \right] s^{\frac{1}{2}} \Delta s = \infty
\]

According to Theorem 3.2, every solution of (39) is oscillatory.

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**Competing Interests**

The author(s) do not have any competing interests in the manuscript.

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