KAUFFMAN BRACKET OF 2- AND 3-STRAND BRAID LINKS

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Abstract. In this paper we give explicit formulas of the Kauffman bracket of the 2-strand braid link $c_{x_1}$ and the 3-strand braid link $c_{x_1}^b_{x_2}$. We also show that the Kauffman bracket of the 3-strand braid link $c_{x_1}^b_{x_2}$ is actually the product of the Kauffman brackets of the 2-strand braid links $c_{x_1}$ and $c_{x_2}^b$. 

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1. Introduction

The Kauffman bracket was introduced by L. H. Kauffman in 1987 in [1].

The Kauffman bracket (polynomial) is actually not a knot invariant because it is not invariant under the first Reidemeister move. However, it has many applications and it can be extended to the popular Jones polynomial, which is invariant under all three Reidemeister moves. In the present work we shall confine ourselves to the Kauffman bracket to avoid from unnecessary length and to leave it for applications. In [2] Nizami et al, computed Khovanov Homology of Braid Links and in [3] gave recursive form of Kauffman Bracket.

This paper is organized as follows: In Section 2 we shall give the basic ideas about knots, braids, and the Kauffman bracket. In Section 3 we shall present the main results.
2. Preliminary Notions

2.1. Links. A link is a disjoint union of circles embedded in $\mathbb{R}^3$. A one-component link is called a knot. Links are usually studied via projecting them on a plan; a projection with extra information of overcrossing and undercrossing is called the link diagram.

Two links are isotopic iff one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result by Reidemeister [4] about the isotopic link diagrams is: Two unoriented links $L_1$ and $L_2$ are equivalent if and only if a diagram of $L_1$ can be transformed into a diagram of $L_2$ by a finite sequence of ambient isotopies of the plane and the local (Reidemeister) moves of the following three types:

$$ R_1 \quad R_2 \quad R_3 $$

The set of all links that are equivalent to a link $L$ is called a class of $L$. By a link $L$ we shall always mean the class of $L$.

The main question of knot theory is Which two links are equivalent and which are not? To address this question one needs a knot invariant, a function that gives one value on all links that belong to a single class and gives different values (but not always) on knots that belong to different classes. The present work is basically concerned with this question.

2.2. Braids. Braids were first studied by Emil Artin in 1925 [5, 6], which play an important role in knot theory, see [7, 8] for detail.

An $n$-strand braid is a set of $n$ non intersecting smooth paths connecting $n$ points on a horizontal plane to $n$ points exactly below them on another horizontal plane in an arbitrary order. The smooth paths are called strands of the braid.

A 2-strand braid

The product $ab$ of two $n$-strand braids is defined by putting $b$ above $a$ and gluing their end points.

A braid with only one crossing is called elementary braid. The $i$th elementary braid $x_i$ on $n$ strands is:
A useful property of elementary braids is that every braid can be written as a product of elementary braids. For instance, the above 2-strand braid is $x_i^3 = (x_i^{-1})(x_i^{-1})(x_i^{-1})$.

The closure of a braid $b$ is the link $\hat{b}$ obtained by connecting the lower ends of $b$ with the corresponding upper ends.

An important result by Alexander [9] connecting knots and braids is: Each link can be represented as the closure of a braid.

**Remark 2.1.** In the last section we shall present all the links as closures of products of elementary braids.

**2.3. The Kauffman Bracket.** The Kauffman bracket was introduced by Kauffman in [10].

Before the definition it is better to understand the two types of splitting of a crossing, the $A$-type and the $B$-type splittings:

In the following, the symbols $\bigcirc$ and $\square$ represent respectively the unknot and the disconnected sum.

**Definition 2.1.** The **Kauffman bracket** is the function $\langle \cdot \rangle : \text{Links} \rightarrow \mathbb{Z}[a, a^{-1}]$ defined by the axioms:

\[
\langle L \rangle = a \langle L_A \rangle + a^{-1} \langle L_B \rangle \\
\langle L \sqcup \bigcirc \rangle = (-a^2 - a^{-2}) \langle L \rangle \\
\langle \bigcirc \rangle = 1
\]

Here $L$, $L_A$, and $L_B$ are three links which are isotopic everywhere except at one crossing where the look as in the figure:

**Proposition 2.2.** The Kauffman polynomial is invariant under second and third Reidemeister moves but not under the first Reidemeister move.
3. Main Results

In this section we shall give the Kauffman bracket of the links $\langle x_1^n \rangle$ and $\langle x_1^n x_2^m \rangle$, and show that $\langle x_1^n x_2^m \rangle = \langle x_1^n \rangle \langle x_2^m \rangle$.

The links $\langle x_1^n \rangle$ fall into two categories, the two-component links when $n$ is even and the one-component links (means knots) when $n$ is odd. When $n$ is even, we have:

**Proposition 3.1.** The Kauffman bracket of the link $\langle x_1^n \rangle$, when $n \geq 2$ is even, is

\[ <\langle x_1^n \rangle> = -a^{3n-2} + a^{3n-6} - a^{3n-10} + a^{3n-14} - \cdots - a^{-n+6} - a^{-n-2}. \]  

**Proof.** We prove it by induction on $n$.

When $n = 2$,

\[ \langle x_1^2 \rangle = \langle x_1 \rangle^2 = a \langle x_1 \rangle + a^{-1} \langle x_1 \rangle \]

\[ = a [a \langle x_1 \rangle + a^{-1} \langle x_1 \rangle] + a^{-1} [a \langle x_1 \rangle + a^{-1} \langle x_1 \rangle] \]

\[ = a [a(-a^2 - a^2) + a^{-1}(1)] + a^{-1} [a(1) + a^{-1}(-a^2 - a^2)] \]

\[ = -a^4 - a^{-4}. \]  

(2)

Similarly, we have

\[ \langle x_1^4 \rangle = -a^{10} + a^6 - a^2 - a^{-6} \]

\[ = -a^{3(4)-2} + a^{3(4)-6} + a^{-2} \langle x_1^2 \rangle \]  

(3)

and

\[ \langle x_1^4 \rangle = -a^{16} + a^{12} - a^8 + a^4 - a^0 - a^{-8} \]

\[ = -a^{3(6)-2} + a^{3(6)-6} + a^{-2} \langle x_1 \rangle. \]  

(4)

We now assume the result holds for $n = k$, that is

\[ \langle x_1^k \rangle = -a^{3k-2} + a^{3k-6} - a^{3k-10} + a^{3k-14} - \cdots - a^{-k-6} - a^{-k-2}. \]  

(5)

Now for $n = k + 1$, we, following Equations 3.3 and 3.4, write

\[ \langle x_1^{k+2} \rangle = -a^{3(k+2)-2} + a^{3(k+2)-6} + a^{-2} \langle x_1^k \rangle \]

\[ = -a^{3(k+2)-2} + a^{3(k+2)-6} + a^{-2} \left[ -a^{3k-2} + a^{3k-6} - a^{3k-10} \right. \]

\[ + a^{3k-14} - \cdots - a^{6-k} - a^{-k-2} \]

\[ = -a^{3(k+2)-2} + a^{3(k+2)-6} - a^{3k-4} + a^{3k-8} - a^{3k-12} + a^{3k-16} \]

\[ - \cdots - a^{4-k} - a^{-k-4} \]

\[ = -a^{3(k+2)-2} + a^{3(k+2)-6} - a^{3(k+2)-10} + a^{3(k+2)-14} - a^{3(k+2)-18} \]

\[ + a^{3(k+2)-22} - \cdots - a^{6-(k+2)} - a^{-(k+2)-2}. \]
This completes the proof. □

**Proposition 3.2.** The Kauffman bracket of the knots $\langle x^4_1 \rangle$, when $n$ is odd, is
\[
\langle x^4_1 \rangle = a^{3n-2} - a^{3n-6} + a^{3n-10} - a^{3n-14} + \cdots - a^{-n+6} - a^{-n-2}. \quad (6)
\]

**Proof.** Similar to the proof of Proposition 3.1 □

**Proposition 3.3.** The Kauffman bracket of the braid link $\langle x^3_1 x^3_2 \rangle$, when $b$ is even, is
\[
\langle x^3_1 x^3_2 \rangle = \sum_{i=1}^{b-1} (-1)^{i+1} a^{6b-i+1} + \sum_{i=1}^{b} (-1)^{i+1} (b-i) a^{2b-i+1} - (b-1)a^b + a^{4-2b} + a^{-2b-4}.
\]

**Proof.** We prove it by induction on $b$.

When $b = 2$, we have
\[
\langle x^3_1 x^3_2 \rangle = a \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] + a^{-1} \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] = a^2 \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] + a \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] + a^{-1} \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] = a^3 \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] + a \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] + a^{-1} \left[ a \langle \emptyset \rangle + a^{-1} \langle \emptyset \rangle \right] = a^4 (-a^2 - a^{-2})^2 + a^2 (-a^2 - a^{-2}) + a^2 (-a^2 - a^{-2}) + (-a^2 - a^{-2})^2 + a^2 (-a^2 - a^{-2}) + (-a^2 - a^{-2})^2 + a^2 (-a^2 - a^{-2}) + (-a^2 - a^{-2})^2 + a^2 (-a^2 - a^{-2}) + a^2 (-a^2 - a^{-2}) + (-a^2 - a^{-2})^2 \]
\[
= a^8 + 2a^{-8} = \left[ a^8 \right] + \left[ 1 \right] + \left[ 0 \right] + \left[ 1 + a^{-8} \right] = \sum_{i=1}^{1} (-1)^{i+1} a^{6(2)-4i} + \sum_{i=1}^{2} (-1)^{i+1} (2-i) a^{2(2)-4i} - (2-2)a^{2(2)} + a^{4-2(2)} + a^{-2(2)-4} \]

as required.

Similarly, we get
\[
\langle x^3_1 x^3_2 \rangle = a^{20} - 2a^{16} + 3a^{12} - 2a^8 + 3a^4 - 2 + 2a^{-4} + a^{-12}, \quad (7)
\]
\[
= \left[ a^{20} - 2a^{16} + 3a^{12} \right] + \left[ 3a^4 - 2 + a^{-4} \right] - 2a^8 + \left[ a^{-4} + a^{-12} \right]
\]
Now for
We now assume the result holds for
Deducting from Equations 3.9 and 3.10, we can write
Similarly,
In order to manage the proof, we reorganize (3.7):
\[
\langle x_1^4 x_2^4 \rangle = \left[ a^4 + 2a^{-4} + a^{-12} \right] - \left[ a^2 \right] + \left[ a^{20} \right] - 2 + [ -2a^{16} + 3a^{12} ]
\]
\[
+ \left[ -2a^8 + 3a^4 \right]
\]
\[
= a^{-4} \left[ \langle x_1^4 x_2^4 \rangle \right] - \sum_{i=1}^{3} i(-1)^{i+1} a^{8-4i} \sum_{i=1}^{1} i(-1)^{i+1} a^{24-4i} - 2
\]
\[
+ \sum_{i=2}^{3} i(-1)^{i+1} a^{24-4i} - 2a^8 + 3a^4
\] (8)
\[
\text{Similarly,}
\]
\[
\langle x_1^6 x_2^6 \rangle = a^{32} - 2a^{28} + 3a^{24} - 4a^{20} + 5a^{16} - 4a^{12} + 5a^8 - 4a^4 + 3 - 2a^{-4}
\]
\[
+ 2a^{-8} + a^{-16}
\]
\[
= a^{-4} \left[ \langle x_1^4 x_2^4 \rangle \right] - \sum_{i=1}^{3} i(-1)^{i+1} a^{20-4i} \sum_{i=1}^{3} i(-1)^{i+1} a^{36-4i} - 2a^4
\]
\[
+ \sum_{i=4}^{5} i(-1)^{i+1} a^{36-4i} - 4a^{12} + 5a^8
\] (9)
\[
\text{Deducting from Equations 3.9 and 3.10, we can write}
\]
\[
\langle x_1^b x_2^b \rangle = a^{-4} \left[ \langle x_1^{b-2} x_2^{b-2} \rangle \right] - \sum_{i=1}^{b-3} i(-1)^{i+1} a^{6b-4i-16} + \sum_{i=1}^{b-3} i(-1)^{i+1} a^{6b-4i}
\]
\[
- 2a^{2b-8} + \sum_{i=b-2}^{b-1} i(-1)^{i+1} a^{6b-4i} - (b-2)a^{2b} + (b-1)a^{2b-4}.
\]
\[
\text{We now assume the result holds for } b = k, \text{ that is}
\]
\[
\langle x_1^{k} x_2^{k} \rangle = \sum_{i=1}^{k-1} i(-1)^{i+1} a^{6k-4i} + \sum_{i=1}^{k} (-1)^{i+1} (k-i)a^{2k-4i} - (k-2)a^{2k}
\]
\[
+ a^{4-2k} + a^{-2k-4}
\] (10)
\[
\text{Now for } b = k + 2, \text{ we have}
\]
\[
\langle x_1^{k+2} x_2^{k+2} \rangle = a^{-4} \left[ \langle x_1^{k+2} x_2^{k+2} \rangle \right] - \sum_{i=1}^{k-1} i(-1)^{i+1} a^{6k-4i-4} + \sum_{i=1}^{k-1} i(-1)^{i+1} a^{6k-4i+12}
\]
\[
- 2a^{2k-4} + \sum_{i=k}^{k+1} i(-1)^{i+1} a^{6k-4i+12} - ka^{2k+4} + (k+1)a^{2k}
\]
The Kauffman bracket of the braid link \( \langle x_1^b x_2^m \rangle \), when \( b \) is odd, is

\[
\langle x_1^b x_2^m \rangle = \sum_{i=1}^{b-1} i(-1)^{i+1} a^{b-4i} + \sum_{i=1}^{m} (-1)^{i+1} (b - i)a^{b-4i} + a^{-4b} + a^{-2b-4}.
\]

**Proof.** Similar to the proof of proposition 3.3. \( \square \)

**Proposition 3.5.** The Kauffman bracket of \( \langle x_1^b x_2^m \rangle \), when \( b > m \geq 2 \), is

\[
\langle x_1^b x_2^m \rangle = \sum_{i=1}^{m-1} (-1)^{b+i-1} a^{3(b+m)-4i} + (-1)^{b+1} (m - 1)a^{3b-m} + m \sum_{i=1}^{b-m-1} (-1)^{b+1-i} a^{3b-m-4i} + (-1)^{m+1} (m - 1)a^{-b+3m} + \sum_{i=1}^{m-2} (-1)^{m+1-i} (m - i)a^{-b+3m-4i} + 2a^{-b-m+4} + a^{-b-m-4}.
\]

**Proof.** We first verify the result for arbitrary \( b \) and \( m = 2 \):

Resolving all \( 2^{4+2} \) crossings as were resolved for \( \langle x_1^2 x_2^2 \rangle \) in Proposition 3.3, we get

\[
\langle x_1^1 x_2^2 \rangle = -a^{11} + a^7 - a^3 + 2a^{-1} + a^{-9}.
\]
Similarly, we get
\[
\langle x_1^4 x_2^2 \rangle = a^{14} - a^{10} + 2a^6 - a^2 + 2a^{-2} + a^{-10} \\
= -a^3(x_1^4 x_2^2) + a^6 + a^2 + 2a^{-2} + a^{-10} + a^{-6}
\] (11)

\[
\langle x_1^5 x_2^2 \rangle = -a^{17} + a^{13} - 2a^9 + 2a^5 - a + 2a^{-3} + a^{-11} \\
= -a^3(x_1^5 x_2^2) + a^5 + a + 2a^{-3} + a^{-11} + a^{-7}
\] (12)

\[
\langle x_1^6 x_2^2 \rangle = a^{20} - a^{16} + 2a^{12} - 2a^8 + 2a^4 - 1 + 2a^{-4} + a^{-12} \\
= -a^3(x_1^6 x_2^2) + a + 1 + 2a^{-4} + a^{-8} + a^{-12}
\] (13)

It follows from (3.11), (3.12), and (3.13) that
\[
\langle x_1^b x_2^2 \rangle = -a^3(x_1^b x_2^2) + a^{-b+10} + a^{-b+6} + 2a^{-b+2} + a^{-b-2} + a^{-b-6}.
\]

Now suppose the result is true for \( b = t \) and \( m = 2 \), that is
\[
\langle x_1^t x_2^2 \rangle = (-1)^{t+2}a^{3t+2} + (-1)^{t+1}a^{3t-2} + 2\sum_{i=1}^{t-3}(-1)^{t+1-i}a^{3t-2-4i}
-a^{-t+6} + 2a^{-t+2} + a^{-t-6}.
\] (14)

For \( b = t + 1 \), we have
\[
\langle x_1^{t+1} x_2^2 \rangle = -a^3(x_1^{t+1} x_2^2) + a^{-t+9} + a^{-t+5} + 2a^{-t+1} + a^{-t-3} + a^{-t-7} \\
= -a^3\left[(-1)^{-t+2}a^{3t+2} + (-1)^{t+1}a^{3t-2} + 2\sum_{i=1}^{t-3}(-1)^{t+1-i}a^{3t-2-4i} - a^{-t+6} + 2a^{-t+2} + a^{-t-6}\right] + a^{-t+9} + a^{-t+5} + 2a^{-t+1} + a^{-t-3} + a^{-t-7} \\
= (-1)^{t+3}a^{3t+5} + (-1)^{t+2}a^{3t+1} + 2\sum_{i=1}^{t-3}(-1)^{t+2-i}a^{3t+1-4i} + a^{-t+9} - 2a^{-t+5} + a^{-t-3} + a^{-t+9} + 2a^{-t+1} + a^{-t-3} + a^{-t-7} \\
= (-1)^{t+3}a^{3t+5} + (-1)^{t+2}a^{3t+1} + 2\sum_{i=1}^{t-3}(-1)^{t+2-i}a^{3t+1-4i} + a^{-t+9} - 2a^{-t+5} + a^{-t+9} + 2a^{-t+1} + a^{-t-3} + a^{-t-7} \\
= (-1)^{(t+1)+3}a^{3(t+1)+2} + (-1)^{(t+1)+1}a^{3(t+1)-2} + 2\sum_{i=1}^{(t+1)-3}(-1)^{(t+1)+1-i}a^{3(t+1)-2-4i} + a^{-t+9} - 2a^{-t+5} + a^{-t+9} + 2a^{-t+1} + a^{-t-3} + a^{-t-7} \\
= (-1)^{(t+1)+3}a^{3(t+1)+2} + (-1)^{(t+1)+1}a^{3(t+1)-2} + 2\sum_{i=1}^{(t+1)-3}(-1)^{(t+1)+1-i}a^{3(t+1)-2-4i} + a^{-t+9} - 2a^{-t+5} + a^{-t+9} + 2a^{-t+1} + a^{-t-3} + a^{-t-7}. 
\]
Similarly, we get
\[ \langle x^b x^2 \rangle = \sum_{i=1}^{2} (-1)^{b+4-i} (i) a^{3b+9-4i} + (-1)^{b+1} 2a^{3b-3} \]
\[ + 3 \sum_{i=1}^{b-4} (-1)^{b+1-i} a^{3b-3-4i} + 2a^{-b+9} \]
\[ - a^{-b+5} + 2a^{-b+1} + a^{-b-7} \]
and
\[ \langle x^b x^4 \rangle = \sum_{i=1}^{3} (-1)^{b+5-i} (i) a^{3b+12-4i} + (-1)^{b+1} 3a^{3b-4} \]
\[ + 4 \sum_{i=1}^{b-5} (-1)^{b+1-i} a^{3b-4-4i} - 3a^{-b+12} \]
\[ + 2 \sum_{i=1}^{2} (-1)^{5-i} (4-i) a^{-b+12-4i} + 2a^{-b} + a^{-b-8}. \]

Now with the assumption that the result is true for an arbitrary \( m \), we have
\[
\langle x^b x^{m+1} \rangle = \sum_{i=1}^{m-1} (-1)^{b+m+2-i} (i) a^{3b-m+3-4i} + (-1)^{b+2} (m-1)a^{3b-m+3} \]
\[ + \sum_{i=1}^{b-m-1} (-1)^{b+2-i} a^{3b-m+3-4i} + (-1)^{m+2} (m-1)a^{-b+3m+3} \]
\[ + \sum_{i=1}^{m-2} (-1)^{m+2-i} (m-i)a^{-b+3m+3-4i} - 2a^{-b-m+7} - a^{-b-m-1} \]
\[ + (-1)^{b} a^{3b-(m+1)+4} + \sum_{i=1}^{b-3} (-1)^{b+1-i} (i) a^{3b-(m+1)-4i} \]
\[ + 2a^{-b-(m+1)+4} + a^{-b-(m+1)} + a^{-b-(m+1)-4} \]
\[ = \sum_{i=1}^{(m+1)-1} (-1)^{b+(m+1)+1-i} (i) a^{3b+(m+1)-4i} \]
which finally, in terms of summation form, is the required result.

\[ + m \sum_{i=1}^{b-m-1} (-1)^{b+2-i} a^{3b-m+3-4i} + (-1)^{m+2} (m-1) a^{-b+3m+3} \\
+ \sum_{i=1}^{m-2} (-1)^{m+2-i} (m-i) a^{-b+3m+3-4i} - 2a^{-b-m+7} \\
+ \sum_{i=1}^{b-3} (-1)^{b+1-i} (i) a^{3b-(m+1)-4i} + 2a^{-b-(m+1)+4} + a^{-b-(m+1)-4} \]

\[ = \sum_{i=1}^{(m+1)-1} (-1)^{b+(m+1)+1-i} (i) a^{3(b+(m+1))-4i} + (-1)^{b+1} ma^{3b-m-1} \\
+ \left[ (-1)^{b+1} ma^{3b-m-1} + (-1)^{b} ma^{3b-m-5} + (-1)^{b-1} ma^{3b-m-9} \\
+ \cdots + (-1)^{m+3} ma^{b+3m+7} + \right] + (-1)^{m+2} (m-1) a^{-b+3m+3} \\
+ \left[ (-1)^{m+1} (m-1) a^{-b+3m-1} + (-1)^{m} (m-1) a^{-b+3m-5} \\
+ (-1)^{m-1} (m-3) a^{-b+3m-9} + \cdots + (-1)^{4} 2a^{-b-m+11} + \right] \\
- 2a^{-b-m+7} \\
+ \left[ \left( (-1)^{b} a^{3b-m-5} + (-1)^{b-1} a^{3b-m-9} + \cdots + (-1)^{m+3} a^{-b+3m+7} \right) \\
+ \left( (-1)^{m+2} a^{-b+3m+3} + (-1)^{m+1} a^{-b+3m-1} + (-1)^{m} a^{-b+3m-5} \\
+ \cdots + (-1)^{4} a^{-b-m+11} \right) \right] + 2a^{-b-(m+1)+4} + a^{-b-(m+1)-4} \]

Now collecting terms of same exponents, we get

\[ = \sum_{i=1}^{(m+1)-1} (-1)^{b+(m+1)+1-i} (i) a^{3(b+(m+1))-4i} + (-1)^{b+1} ma^{3b-m-1} \\
+ \left[ (-1)^{b} (m+1) a^{3b-m-5} + (-1)^{b-1} (m+1) a^{3b-m-9} \\
+ \cdots + (-1)^{m+3} (m+1) a^{-b+3m+7} + \right] + (-1)^{m+2} (m) a^{-b+3m+3} \\
+ \left[ (-1)^{m+1} (m) a^{-b+3m-1} + (-1)^{m} (m-1) a^{-b+3m-5} \\
+ \cdots + (-1)^{4} 3a^{-b-m+11} - 2a^{-b-m+7} \right] \\
+ 2a^{-b-(m+1)+4} + a^{-b-(m+1)-4} \]

which finally, in terms of summation form, is the required result. \( \square \)
Theorem 3.6. For any $b, m \geq 2$, 
$$
\langle x_1^b x_2^m \rangle = \langle x_1^b \rangle \langle x_1^m \rangle.
$$

Proof. Depending on $b$ and $m$, the proof is divided into three cases: when $b, m$ are even and equal, when $b, m$ are odd and equal, and when $b, m$ are distinct.

Case I. (When $b$ and $m$ are even and equal.)

In this case, letting $m = b$, we show that $\langle x_1^b x_2^m \rangle = \langle x_1^b \rangle \langle x_1^m \rangle$. So, we proceed as follows:

$$
\langle x_1^2 x_2^2 \rangle = a^8 + 2 + a^{-8} = (-a^4 - a^{-4})(-a^4 - a^{-4}) = \langle x_1^2 \rangle \langle x_2^2 \rangle.
$$

Also, we have

$$
\langle x_1^4 x_2^4 \rangle = a^{20} - 2a^{16} + 3a^{12} - 2a^8 + 3a^4 - 2 + 2a^{-4} + a^{-12}
$$

$$
= (-a^{10} + a^6 - a^2 - a^{-6})(-a^{10} + a^6 - a^2 - a^{-6})
$$

$$
= \langle x_1^4 \rangle \langle x_1^4 \rangle
$$

and

$$
\langle x_1^6 x_2^6 \rangle = a^{32} - 2a^{28} + 3a^{24} - 4a^{20} + 5a^{16} - 4a^{12} + 5a^8 - 4a^4 + 3 - 2a^{-4} + 2a^{-8} + a^{-16}
$$

$$
= (-a^{16} + a^{12} - a^8 + a^4 - a^0 - a^{-8})(-a^{16} + a^{12} - a^8 + a^4 - a^0 - a^{-8})
$$

$$
= \langle x_1^6 \rangle \langle x_1^6 \rangle.
$$

Now we assume that the result is true for $b = k$, that is

$$
\langle x_1^k x_2^k \rangle = \langle x_1^k \rangle \langle x_1^k \rangle.
$$

Since $\langle x_1^n \rangle = -a^{3(n)-2} + a^{3(n)-6} + a^{-2(\langle x_1^{n-2} \rangle)}$, we have

$$
\langle x_1^{k+2} \rangle \langle x_1^{k+2} \rangle = \left[ -a^{3k+4} + a^{3k} + a^{-2(\langle x_1^k \rangle)} \right] \left[ -a^{3k+4} + a^{3k} + a^{-2(\langle x_1^k \rangle)} \right]
$$

$$
+ a^{-2(\langle x_1^k \rangle)} \langle x_1^{k+2} \rangle
$$

$$
= a^{-4} \left[ \langle x_1^k \rangle \right]^2 + a^{6k+8} - 2a^{6k+4} + a^{6k} - 2a^{3k+2} \langle x_1^k \rangle
$$

$$
+ 2a^{3k-2} \langle x_1^k \rangle
$$

$$
= a^{-4} \left[ \langle x_1^k \rangle \right]^2 + a^{6k+8} - 2a^{6k+4} + a^{6k} + 2a^{6k} - 2a^{6k-4} + 2a^{6k-8} - 2a^{6k-12} + \cdots + 2a^{2k+12} + 2a^{2k+8} + 2a^{2k} - 2a^{6k-4} + 2a^{6k-8} - 2a^{6k-12} + 2a^{6k-16} - \cdots + 2a^{2k+8} - 2a^{2k+4} - 2a^{2k-4}
$$

$$
= a^{-4} \left[ \langle x_1^k \rangle \right]^2 + a^{6k+8} - 2a^{6k+4} + a^{6k} - 4a^{6k-4} + 4a^{6k-8} - 4a^{6k-12} + 4a^{6k-16} - \cdots + 4a^{2k+8} - 2a^{2k+4} + 2a^{2k} - 2a^{2k-4}.
$$

(15)
Also
\[
\langle x_1^{k+2} x_2^{k+2} \rangle = a^{-4} \left[ \langle x_1^k x_2^k \rangle \right] - \sum_{i=1}^{k-1} i(-1)^{i+1} a^{6k-4i-4} \\
+ \sum_{i=1}^{k-1} i(-1)^{i+1} a^{6k-4i+12} - 2a^{2k-4} \\
+ \sum_{i=k}^{k+1} i(-1)^{i+1} a^{6k-4i+12} - ka^{2k+4} + (k+1)a^{2k} \\
= a^{-4} \left[ \langle x_1^k \rangle \right]^2 - a^{6k-8} + 2a^{6k-12} - 3a^{6k-16} + 4a^{6k-20} \\
- \ldots - (k-3)(-1)^{k-2} a^{2k+8} - (k-2)(-1)^{k-1} a^{2k+4} \\
- (k-1)(-1)^{k} a^{2k} + a^{6k+8} - 2a^{6k+4} + 3a^{6k} - 4a^{6k-4} \\
+ 5a^{6k-8} - 6a^{6k-12} + 7a^{6k-16} - 8a^{6k-20} + \ldots \\
+ (k-3)(-1)^{k-2} a^{2k+24} + (k-2)(-1)^{k-1} a^{2k+20} \\
+ (k-1)(-1)^{k} a^{2k+16} - 2a^{2k-4} + k(-1)^{k+1} a^{2k+12} \\
+ (k+1)(-1)^{k+1} a^{2k+8} - ka^{2k+4} + (k+1)a^{2k} \\
= a^{-4} \left[ \langle x_1^k \rangle \right]^2 + a^{6k+8} - 2a^{6k+4} + 3a^{6k} - 4a^{6k-4} + 4a^{6k-8} \\
- 4a^{6k-12} + 4a^{6k-16} - \ldots + 4a^{2k+8} - 2a^{2k+4} + 2a^{2k} \\
- 2a^{2k-4}. \quad (16)
\]

The result now follows from (3.15) and (3.16).

**Case II. (When b and m are odd and equal.)** Similar to Case I.

**Case III. (When b and m are distinct.)**

In order to prove this part let us agree on the terminology:
\[
\begin{align*}
\pi_n &= (-1)^{n+m} a^{4n-4m-2}, \quad n = 1, 2, \ldots, m-1, \pi_m = -a^{-m-2} \\
\bar{y}_l &= (-1)^{l+b} a^{3b-4l-2}, \quad l = 1, 2, \ldots, b-1, \bar{y}_b = -a^{-b-2} \\
i &= 1, 2, \ldots, m, \quad j = 1, 2, \ldots, b; b \geq 2
\end{align*}
\]

\[
\langle x_i^j \rangle \langle x_i^j \rangle = \sum_{i+j=2} \pi_i \bar{y}_j + \sum_{i+j=m+1, i \neq m} \pi_i \bar{y}_j + \left[ \sum_{i+j=m+2, i \neq m} \pi_i \bar{y}_j + \pi_m \bar{y}_1 \right] \\
+ \sum_{i+j=m+1, i \neq m} \pi_i \bar{y}_j + \pi_m \bar{y}_2 + \ldots + \sum_{i+j=b, i \neq m} \pi_i \bar{y}_j + \pi_m \bar{y}_{b-1},
\]

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\[
\sum_{i+j=b+1, i \neq 1, m} x_i y_j + \sum_{i+j=b+2, i \neq 2, m} x_i y_j + \sum_{i+j=b+3, i \neq 3, m} x_i y_j + \cdots \\
+ \left( \sum_{i+j=b+3, i \neq 3, m} x_i y_j + x_m y_{b-m+2} + x_2 y_b \right) + \cdots \\
+ \left( \sum_{i+j=b+m-3, i \neq m-3, m} x_i y_j + x_m y_{b-4} + x_{m-4} y_b \right) \\
+ \left( x_m y_{b-1} + x_m y_{b-3} + x_m y_{b-5} \right) + \left( x_m y_{b-2} + x_m y_{b-4} \right) \\
+ \left( x_m y_{b-1} + x_m y_{b-3} \right) + x_m y_b
\]

Since this agrees with the result of Proposition 3.5, the proof is finished. \qed

Competing Interests

The author(s) do not have any competing interests in the manuscript.

References


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